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Leslie Hogben

Iowa State University, hogben@aimath.org

Clifford Bergman

Iowa State University, cbergman@iastate.edu

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Abstract

A variety of universal algebras is called deductive if every subquasivariety is a variety. The following results are obtained: (1) The variety of modules of an Artinian ring is deductive if and only if the ring is the direct sum of matrix rings over local rings, in which the maximal ideal is principal as a left and right ideal. (2) A directly representable variety of finite type is deductive if and only if either (i) it is equationally complete, or (ii) every algebra has an idempotent element, and a ring constructed from the variety is of the form (1) above.

Keywords

subquasivariety, Artinian ring, idempotent element

Disciplines

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Comments

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DEDUCTIVE VARIETIES OF MODULES AND UNIVERSAL ALGEBRAS

BY

LESLIE HOGBEN AND CLIFFORD BERGMAN¹

ABSTRACT. A variety of universal algebras is called deductive if every subquasi-variety is a variety. The following results are obtained: (1) The variety of modules of an Artinian ring is deductive if and only if the ring is the direct sum of matrix rings over local rings, in which the maximal ideal is principal as a left and right ideal. (2) A directly representable variety of finite type is deductive if and only if either (i) it is equationally complete, or (ii) every algebra has an idempotent element, and a ring constructed from the variety is of the form (1) above.

This paper initiates a study of varieties of universal algebras with the property that every subquasivariety is a variety. We call such varieties *deductive*. In §2 it is proved that the variety of left modules of a left Artinian ring R is deductive if and only if R is the direct sum of matrix rings over local rings in which the maximal ideal is principal as a left and right ideal. In §3 we consider varieties of universal algebras of finite type that are directly representable. A variety is directly representable if it is generated by a finite algebra and has, up to isomorphism, only finitely many directly indecomposable finite algebras. Such a variety is deductive if and only if either (i) it is equationally complete, or (ii) every algebra has an idempotent element and a certain ring, constructed from the variety, has the properties listed above.

Deductive varieties were first considered by those studying the varieties arising in algebraic logic. (The Russian equivalent for “deductive” would be “hereditarily structurally complete, in the finitary sense.”) Roughly speaking, subvarieties correspond to new axioms added to the logic under consideration, while subquasivarieties correspond to new rules of inference. Thus, in the logic associated with a deductive variety, every rule of inference can be replaced by a set of logical axioms. Some other results on deductive varieties can be found in Tsitkin [19] on varieties of Heyting algebras, and in Igošin [10] on some deductive varieties of lattices.

This paper got its start with the consideration of discriminator varieties, which falls under the heading of algebraic logic. From there the material in §3 developed, reducing the problem of characterizing those directly representable varieties that are deductive to the corresponding problem for finite rings.

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1. Preliminaries. Throughout this paper, an algebra \mathbf{A} is an object $(A, F_i)_{i \in I}$, in which A is a nonempty set and each F_i is a function from A^{n_i} to A , n_i a natural number. The set $\{F_i: i \in I\}$ is the set of *basic operations* of \mathbf{A} , and the sequence $\langle n_i: i \in I \rangle$ is called the *type* of \mathbf{A} . If I is finite, then \mathbf{A} is of *finite type*.

A *term* of an algebra \mathbf{A} is an operation built from the basic operations by composition. An identity is an expression of the form

$$\forall x_1, \dots, x_n (\sigma(x_1, \dots, x_n) = \tau(x_1, \dots, x_n))$$

in which σ and τ are n -ary terms. A *quasi-identity* is either an identity or an expression of the form

$$\forall \bar{x} [(\sigma_1 = \tau_1 \ \& \ \sigma_2 = \tau_2 \ \& \ \cdots \ \& \ \sigma_k = \tau_k) \rightarrow \mu = \nu],$$

in which $\sigma_i, \tau_i, \mu, \nu$ are all terms.

A *variety* of algebras (of the same type) is a class of all algebras satisfying a fixed set of identities, and a *quasivariety* is a class of algebras satisfying a fixed set of quasi-identities. A fundamental theorem of universal algebra states that a class of algebras is a variety if and only if it is closed under the formation of homomorphic images, products and subalgebras. As every quasivariety is closed under the latter two constructions, we conclude that a variety is deductive if and only if every subquasivariety is closed under the formation of homomorphic images.

Since we will be discussing the variety of modules over a ring extensively, we describe it explicitly. By a *ring* we mean an associative ring R with multiplicative identity element 1_R . (We do not require a subring of R to have the same identity element as R itself.) The variety of left R -modules, $R\text{-mod}$, is the class of all algebras $(M, +, -, 0, l_r)_{r \in R}$ such that $(M, +, -, 0)$ is an abelian group, l_r is an endomorphism of this group (for each $r \in R$); and $l_1(m) = m$ (for all $m \in M$, where $1 = 1_R$), $l_{r+s} = l_r + l_s$, and $l_r \circ l_s = l_{rs}$ (for all $r, s \in R$). By an R -module, we mean a member of $R\text{-mod}$, more commonly called a left unital R -module. In practice, we write rm for $l_r(m)$.

Every quasi-identity for the class $R\text{-Mod}$ can be equivalently expressed in the form

$$\forall x_1, \dots, x_n \left[\bigwedge_{i=1}^k \left(\sum_{j=1}^n r_{ij} x_j = 0 \right) \rightarrow \sum_{j=1}^n s_j x_j = 0 \right].$$

2. Rings whose varieties of modules are deductive. We shall call a ring deductive iff the variety $R\text{-Mod}$ is deductive. Artinian will mean left Artinian.

2.1. EXAMPLE. Any semisimple Artinian ring is deductive because every module of a semisimple Artinian ring is projective. Therefore any homomorphic image of a module M over such a ring is isomorphic to a direct summand of M . Note that a semisimple Artinian ring is a finite direct sum of matrix rings over division rings.

2.2. EXAMPLE. \mathbf{Z} = the integers is not deductive because the cyclic module \mathbf{Z} is in the quasivariety defined by

$$\forall x (2x = 0 \rightarrow x = 0),$$

but $\mathbf{Z}/2\mathbf{Z}$ is not. Note that the ring \mathbf{Z} is semisimple but not Artinian.

Similarly, any integral domain which is not a field is not deductive. This example shows that a subring of a deductive ring need not be deductive. (\mathbf{Z} is a subring of the rationals.)

2.3. EXAMPLE. Let $F[x, y]$ be the ring of polynomials in the (commuting) variables x and y over a field F . Let R be the factor ring of $F[x, y]$ by the ideal generated by x^2, y^2 and xy . That is, $R = F[x, y]/(x^2, y^2, xy)$. Denoting the images of 1, x and y in R by 1, a and b , respectively, $\{1, a, b\}$ is a basis for R over F . We have $a^2 = 0$, $b^2 = 0$ and $ab = 0$. The radical of R is $N = Fa + Fb$, and $N^2 = 0$. R is Artinian (if F is finite so is R). R is not deductive: R is in the quasivariety defined by the quasi-identity $\forall x (ax = 0 \rightarrow bx = 0)$, since $ax = 0$ iff $x \in N$ iff $bx = 0$. R/Fa is not in this quasivariety, since $a(1 + Fa) = 0 + Fa$ but $b(1 + Fa) \neq 0 + Fa$. This example shows that not all finite rings are deductive.

If R is deductive then so is every homomorphic image $f(R)$, since any $f(R)$ -module is also an R -module.

Decomposition of rings. A ring is called *indecomposable* if it cannot be expressed as the direct sum of two nonzero ideals. Note that when a ring decomposes as a finite direct sum, $R = \bigoplus_{i=1}^q R_i$, each ideal R_i is a ring with identity element 1^i where $1 = \bigoplus 1^i$. In this case, any R -module M decomposes as $M = \bigoplus_i M_i$, where $M_i = 1^i M$. M_i is both an R_i - and an R -module.

Any Artinian ring can be expressed as the finite direct sum of indecomposable rings, which allows us to reduce our study of deductive Artinian rings to deductive indecomposable Artinian rings:

2.4. THEOREM. Let $R = \bigoplus_{i=1}^q R_i$ (direct sum as rings). Then R is deductive if and only if R_i is deductive for all $i = 1, \dots, q$. An Artinian ring is deductive if and only if each of its indecomposable direct summands is deductive.

PROOF. Since a direct summand is a homomorphic image, if R_i is not deductive, neither is R .

Since a quasivariety defined by several quasi-identities is the intersection of the quasivarieties defined by the single quasi-identities, it suffices to consider quasivarieties defined by single quasi-identities.

With the quasivariety \mathcal{Q} (of R -modules) defined by

$$\forall x_1, \dots, x_n \left[\bigwedge_{i=1}^k \left(\sum_{j=1}^n r_{ij} x_j = 0 \right) \rightarrow \sum_{j=1}^n s_j x_j = 0 \right],$$

we associate the quasivarieties \mathcal{Q}_i defined by

$$\forall x_1, \dots, x_n \left[\bigwedge_{i=1}^k \left(\sum_{j=1}^n r_{ij}^i x_j = 0 \right) \rightarrow \sum_{j=1}^n s_j^i x_j = 0 \right].$$

An R -module M is in \mathcal{Q} iff M_t is in \mathcal{Q}_t for all t . The R -module M_t is in \mathcal{Q}_t iff the R_t -module M_t is in \mathcal{Q}_t . Let M be in \mathcal{Q} and let $f(M)$ be a homomorphic image of M . M_t is in \mathcal{Q}_t (for all t). Since R_t is deductive, $f(M_t)$ (which is an R_t -image) is in \mathcal{Q}_t (for all t). $f(M_t) = f(M)_t$. Therefore $f(M) \in \mathcal{Q}$. So if R_t is deductive for all t then so is R . \square

Example 2.3 shows the importance of the assumption that all rings have 1 in this theorem, since R can be constructed by taking the direct sum of Fa and Fb and adjoining 1. Fa is a rng (ring without 1), but by adjoining a 1 and applying Theorem 2.15 we could show Fa is a deductive rng. Similarly, Fb is deductive. But $Fa \oplus Fb$, as Example 2.3 shows, is not.

Indecomposable Artinian rings. Theorem 2.4 shows that the building blocks of deductive Artinian rings are the deductive indecomposable Artinian rings. We next show that a deductive indecomposable Artinian ring is a matrix ring over a local ring. Recall that a ring R is local iff $R/J(R)$ is a division ring, where $J(R)$ is the Jacobson radical of R . (Local rings are discussed in more detail later.)

2.5. THEOREM. *A deductive indecomposable Artinian ring is a matrix ring over a local ring.*

The technique that will be used to prove this theorem is derived from the following example, where it is unobscured by detail.

2.6. EXAMPLE. Let F be a field. Let R be a vector space over F with basis $\{e, f, n\}$. An F -linear multiplication in R is defined by setting $ee = e$, $ff = f$, $en = n$, $nf = n$, and taking all other products of basis elements to be 0. This makes R into a ring, with $1 = e + f$ and $J(R) = Fn$. R is indecomposable but $R/J(R) = F \oplus F$ (as a ring). The quasivariety \mathcal{Q} defined by

$$\forall x [(ex = 0 \ \& \ nx = 0) \rightarrow fx = 0]$$

is not a variety because $R \in \mathcal{Q}$, but $R/(Fe + Fn) \notin \mathcal{Q}$.

PROOF OF THEOREM 2.5. Let R be a deductive indecomposable Artinian ring. Let N be its radical. R/N is a semisimple Artinian ring, so R/N is the direct sum of matrix rings over division rings. We shall prove that R/N is directly indecomposable. From this it will follow that R/N is isomorphic to a matrix ring $M_n(D)$ over a division ring D . Now N is nilpotent, and so idempotents can be lifted modulo the radical, and it will follow from [11, Theorem 1, p. 55] that $R \cong M_n(S)$ where $S/J(S) \cong D$. Thus the desired result follows from the direct indecomposability of R/N .

To derive a contradiction, suppose that R/N is not directly indecomposable. Then it contains two nonzero orthogonal central idempotents whose sum is 1. From [11, Theorem 2, p. 56] we have the following results: lifting these idempotents we obtain nonzero orthogonal idempotents e and f in R such that $e + f = 1$ and e, f are central modulo N . Thus $R = eRe + fRf + eRf + fRe$. This is a direct sum of additive groups. Since e and f are orthogonal and central modulo N , $eRf + fRe \subseteq N$.

By [1, Corollary 27.18, p. 311], R/N^2 is indecomposable. Since R is deductive, so is R/N^2 . Thus all of our assumptions about R are valid for R/N^2 . We may replace R by R/N^2 , or equivalently, assume that $N^2 = (0)$.

Since R is indecomposable, at least one of eRf, fRe is nonzero.

Assume $eRf \neq 0$. Since $N^2 = 0$, eRf is an ideal. Since R is Artinian, R is Noetherian and eRf is finitely generated. Let n_1, \dots, n_p generate eRf as a left ideal. Define the quasivariety \mathcal{Q} by

$$\forall x \left[\left(ex = 0 \ \& \ \bigwedge_i n_i x = 0 \right) \rightarrow fx = 0 \right].$$

For subsets S and K of R we define the right annihilator of S modulo K as $r(S, K) = \{t \in R : St \subseteq K\}$, and we put $r(S) = r(S, 0)$. If S is a right ideal and K closed under addition then $r(S, K)$ is a left ideal. If K is a right ideal then $r(S, K)$ is a right ideal. Note that $r(eRf) = r(\{n_1, \dots, n_p\})$.

We construct a left ideal K such that $R/K \in \mathcal{Q}$ but $f \notin eRe + N + K$. Let $K = fRe + (r(eRf) \cap fRf)$. Since $N^2 = 0$, fRe and eRf are ideals and therefore so is $r(eRf)$. K is a left ideal because fRe is an ideal and $r(eRf) \cap fRf$ is a left ideal:

$$(eRf)r(eRf) = 0 \quad \text{and} \quad (eRe)(fRf) = 0;$$

so

$$(eR)(r(eRf) \cap fRf) = (eRe + eRf)(r(eRf) \cap fRf) = 0.$$

$(fR)fRf \subseteq fRf$ and $(fR)r(eRf) \subseteq r(eRf)$. Thus

$$R(r(eRf) \cap fRf) = (eR + fR)(r(eRf) \cap fRf) \subseteq r(eRf) \cap fRf;$$

so $r(eRf) \cap fRf$ is a left ideal. If $f \in eRe + N + K$ then $f \in N + K \subseteq r(eRf)$, but $(eRf)f = eRf \neq 0$, so $f \notin eRe + N + K$. Finally, $r(eRf, K) \cap fRf \subseteq K$, because if $t \in r(eRf, K) \cap fRf$, then $(eRf)t \subseteq K$, $(eRf)t \subseteq eR \cap K \subseteq eR \cap fR = 0$; so $t \in r(eRf)$. Thus

$$r(eRf, K) \cap fRf \subseteq r(eRf) \cap fRf \quad \text{and} \quad r(eRf) \cap fRf \subseteq K.$$

We show R/K is in \mathcal{Q} : Let $t \in R$. If $et = 0 \pmod{K}$, then $t = ft \pmod{K}$. Since $fRe \subseteq K$, $t = ftf \pmod{K}$. If $n_i t = 0 \pmod{K}$ for all $i = 1, \dots, p$, then $ftf \in r(eRf, K) \cap fRf \subseteq K$. Thus $t = 0 \pmod{K}$; so $ft = 0 \pmod{K}$ and $R/K \in \mathcal{Q}$. $eRe + N + K$ is a left ideal, because $R = eRe + fRf + N$ and $fRfeRe = 0$. But $R/(eRe + N + K) \notin \mathcal{Q}$ because the hypotheses are always true, but $ff = f \notin eRe + N + K$.

The preceding paragraph contradicts the fact that R is deductive. This concludes our proof. \square

Rings whose finitely generated modules are direct sums of cyclic modules. Rings with the property that every finitely generated module can be decomposed into the direct sum of cyclic modules will be referred to as FGC rings. In this section, we shall show, in Theorem 2.8, that if R is an FGC ring and \mathcal{Q} is a subquasivariety of $R\text{-Mod}$, then \mathcal{Q} is a variety iff the class of cyclic R -modules in \mathcal{Q} is closed under homomorphisms.

We make some preliminary observations. For any ring R and quasivariety $\mathcal{Q} \subseteq R\text{-Mod}$, if $M = \prod_i M_i$ (or $M = \bigoplus_i M_i$), then $M \in \mathcal{Q}$ iff $M_i \in \mathcal{Q}$ for all i .

Let M be an R -module in \mathcal{Q} . To show that \mathcal{Q} is a variety we must show that any homomorphic image of M is in \mathcal{Q} . An R -module is in \mathcal{Q} iff every finitely generated submodule is in \mathcal{Q} . A finitely generated submodule of a homomorphic image of M is

a homomorphic image of a finitely generated submodule of M . So without loss of generality we may assume that M is finitely generated.

We need the following

2.7. LEMMA. Let R be a ring. Let \mathcal{Q} be a quasivariety of R -modules. Let M be an R -module and L_i ($i \in I$) be left ideals of R . Then:

- (1) If $R/L_i \in \mathcal{Q}$ for all i , then $R/(\bigcap_{i \in I} L_i) \in \mathcal{Q}$.
- (2) If $M \in \mathcal{Q}$, then $R/\text{ann}(M) \in \mathcal{Q}$. ($\text{ann}(M) = \{r \in R: rM = 0\}$).

PROOF. Quasivarieties are closed under submodules and direct products. For (1) define $f: R \rightarrow \prod_i R/L_i$ by $f(r)_i = r + L_i$. f is a homomorphism and its kernel is $\bigcap_i L_i$. Thus $R/(\bigcap_i L_i)$ is isomorphic to a submodule of a product of elements of \mathcal{Q} and so $R/(\bigcap_i L_i) \in \mathcal{Q}$.

For (2) note that if $M \in \mathcal{Q}$ then $Rm \in \mathcal{Q}$ for any $m \in M$. The cyclic R -module Rm is isomorphic to $R/\text{ann}(m)$, where $\text{ann}(m) = \{r \in R: rm = 0\}$. Therefore $R/\text{ann}(m) \in \mathcal{Q}$ for all $m \in M$. $\text{ann}(M) = \bigcap_{m \in M} \text{ann}(m)$ and, by (1), $R/\text{ann}(M) \in \mathcal{Q}$. \square

2.8. THEOREM. Let R be an FGC ring. Let \mathcal{Q} be a quasivariety of R -modules. \mathcal{Q} is a variety iff $R/I \in \mathcal{Q}$ implies $R/L \in \mathcal{Q}$, for any ideal I and left ideal L of R with $I \subseteq L$.

PROOF. Let $M \in \mathcal{Q}$ (M finitely generated) and let $f(M)$ be a homomorphic image of M . By Lemma 2.7, $R/\text{ann}(M) \in \mathcal{Q}$. (Note that $\text{ann}(M)$ is an ideal of R .)

Since M is finitely generated so is $f(M)$, and therefore $f(M)$ is isomorphic to a direct sum of cyclic modules Rp_j . Rp_j is isomorphic to $R/\text{ann}(p_j)$. Since the annihilator of M is contained in the annihilator of $f(M)$, $\text{ann}(M) \subseteq \text{ann}(p_j)$ for all j . Thus by our hypothesis, $R/\text{ann}(p_j) \in \mathcal{Q}$ for all j , so $f(M) \in \mathcal{Q}$. Thus \mathcal{Q} is a variety. \square

Example 2.1 (the integers) shows that not all FGC rings are deductive. If R is Artinian, then any finitely generated R -module M can be expressed as the finite direct sum of indecomposable submodules. Thus an Artinian ring is an FGC ring if and only if every indecomposable module is cyclic. Semisimple Artinian rings are FGC rings.

Matrix rings. Having reduced the question of which Artinian rings are deductive to the question of which Artinian matrix rings over local rings are deductive, we now consider the relationship between the deductiveness of R and the deductiveness of $M_n(R)$ (the complete matrix ring over R):

2.9. THEOREM. If $M_n(R)$ is deductive, then so is R . If R is a deductive FGC ring, then so is $M_n(R)$.

PROOF. Let $\{e_{ab}\}$ be matrix units (e_{ab} has 1 in the ab place, 0 everywhere else).

Given any R -module M we can construct an $M_n(R)$ -module M^n (the direct sum of n copies of M). We can extend any R -homomorphism f of M to an $M_n(R)$ -homomorphism of M^n in the obvious way:

$$f(m_1, \dots, m_n) = (f(m_1), \dots, f(m_n)).$$

Given any R -quasi-identity q we associate an $M_n(R)$ -quasi-identity \tilde{q} by replacing an element r of R by re_{11} . M satisfies q iff M^n satisfies \tilde{q} . Thus if R is not deductive, then neither is $M_n(R)$.

Now assume R is a deductive FGC ring. The categories of R -modules and $M_n(R)$ -modules are equivalent [12, Proposition 1.4, p. 29 and Morita I, pp. 166–168]. The functor used to go from $R\text{-Mod}$ to $M_n(R)\text{-Mod}$, $R^n \otimes_R -$ maps an R -module M to M^n (as described above). If M is the direct sum of cyclic submodules, then so is M^n (as an $M_n(R)$ -module). So for any finitely generated $M_n(R)$ -module we map by the functor ${}^nR \otimes_{M_n(R)} -$ to an R -module (which will also be finitely generated), decompose this module into the direct sum of cyclic submodules, map back by the other functor to an $M_n(R)$ -module which is also the direct sum of cyclics. Since these functors constitute an equivalence of categories, the decomposed module is isomorphic to the original. Thus $M_n(R)$ is an FGC ring. Therefore by Theorem 2.8 it suffices to show that if $M_n(R)/I \in \mathcal{Q}$, then $M_n(R)/L \in \mathcal{Q}$ (I an ideal, L a left ideal of $M_n(R)$, $I \subseteq L$).

We may assume that \mathcal{Q} is defined by the single quasi-identity

$$\forall x_1, \dots, x_m \left[\bigwedge_{i=1}^k \left(\sum_{j=1}^m r_{ij} x_j = 0 \right) \rightarrow \sum_{j=1}^m s_j x_j = 0 \right].$$

Remember these are in $M_n(R)$: Let

$$r_{ij} = \sum_{a,b} r_{ij}^{ab} e_{ab}, s_j = \sum_{a,b} s_j^{ab} e_{ab}, x_j = \sum_{a,b} x_j^{ab} e_{ab}.$$

Define a quasivariety of R -modules $\tilde{\mathcal{Q}}$ by the sentence

$$\forall x_j^{bc} \left[\bigwedge_{i=1}^k \bigwedge_{a=1}^n \bigwedge_{c=1}^n \left(\sum_{j=1}^m \sum_{b=1}^n r_{ij}^{ab} x_j^{bc} = 0 \right) \rightarrow \bigwedge_{a=1}^n \bigwedge_{c=1}^n \left(\sum_{j=1}^m \sum_{b=1}^n s_j^{ab} x_j^{bc} = 0 \right) \right].$$

It is easy to verify that this sentence is equivalent to a set of quasi-identities. An $M_n(R)$ -module $M_n(R)/L$ is in \mathcal{Q} if and only if $M_n(R)/L$, regarded as an R -module, is in $\tilde{\mathcal{Q}}$ (with x_j corresponding to x_j^{bc}). Since R is deductive, \mathcal{Q} is closed under homomorphic images, so if $M_n(R)/I \in \mathcal{Q}$ and $I \subseteq L$, $M_n(R)/I \in \tilde{\mathcal{Q}}$, so $M_n(R)/L \in \tilde{\mathcal{Q}}$, so $M_n(R)/L \in \mathcal{Q}$. Therefore $M_n(R)$ is deductive. \square

Local rings. To complete the classification of deductive Artinian rings we need to determine which local rings are deductive.

Recall that a ring R is called *local* if the following three equivalent conditions hold: (1) the set of nonunits is an ideal, (2) there is a unique maximal left ideal, and (3) $R/J(R)$ is a division ring [1, Proposition 15.15, p. 170].

2.10. LEMMA. *Let R be a ring. The following are equivalent:*

- (1) R is Artinian and local, and its maximal ideal N is principal as a left ideal.
- (2) R has only finitely many left ideals and these form a chain.
- (3) There is an element d of R such that $d^k = 0$ for some integer k , and every left ideal is of the form Rd^i for some i (defining $d^0 = 1$).

PROOF. (3) \Rightarrow (2) is clear. Since a linear ordering of left ideals implies every left ideal is principal, (2) \Rightarrow (1).

(1) \Rightarrow (3). Let $N = Rd$ be the maximal ideal of R . Since R is Artinian, the radical N is nilpotent, and $d^k = 0$ for some k . Let $L \subseteq N$ be any left ideal. If $L \cap (Rd^i - Rd^{i+1}) \neq \emptyset$, then $Rd^i \subseteq L$: Let $x \in L \cap (Rd^i - Rd^{i+1})$. $x = rd^i$. Since $x \notin Rd^{i+1}$, $r \notin N$; so r is invertible and $Rd^i \subseteq L$. So $L = Rd^j$ for j the least such that $L \cap (Rd^j - Rd^{j+1}) \neq \emptyset$. \square

A ring R satisfying the conditions of Lemma 2.10 is called a left finite chain ring. Note that in a left finite chain ring every left ideal is an ideal. The left annihilator of d^i , $l(d^i)$, is Rd^{k-i} . A right finite chain ring is defined analogously. A left finite chain ring is a right finite chain ring if and only if $Rd = dR$.

A ring which is both a left finite chain ring and a right finite chain ring will be called an LRFCR. Such rings are the basic building blocks of deductive Artinian rings.

2.11. EXAMPLE. If R is a principal ideal domain and P is a prime ideal of R , then $R/(P^n)$ is an LRFCR. The commutative LRFCRs are precisely the proper homomorphic images of complete discrete valuation rings [15].

Commutative LRFCRs which are finite are of the form $T[x]/(f(x), p^s x')$, where T is a Galois ring and f is an Eisenstein polynomial. Isomorphism classes of such rings have been enumerated [4].

2.12. EXAMPLE. If D is a division ring and $R = D[x]/(D[x]x^n)$, then R is an LRFCR.

It is possible for a ring to be a left finite chain ring but not a right finite chain ring, as the following example shows.

2.13. EXAMPLE. Let S be a twisted polynomial ring constructed as follows: Let F be a (infinite) field with a monomorphism $\alpha \rightarrow \hat{\alpha}$ which is not surjective. S is the set of polynomials in an indeterminate x over F with coefficients written on the left, i. e. $p(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0$ with the usual addition. Products are defined by the distributive law, operations of F , and $x\alpha = \hat{\alpha}x$. S is a left Euclidean domain [5]. Sx^2 is an ideal of S . Let R be the factor ring S/Sx^2 . R is a left but not a right finite chain ring. R is not deductive and, in fact, was the prototype for the argument given below that a left finite chain ring which is not right chain is not deductive.

Let us continue with the classification. In order to use Theorem 2.8 we need the following

2.14. LEMMA. *An LRFCR is an FGC ring.*

Standard methods of proof for modules over a PID work for LRFCRs. See, for example, [9, p. 295].

2.15. THEOREM. *An Artinian local ring is deductive if and only if it is a left and right finite chain ring.*

PROOF. Let R be an LRFCR. To show that R is deductive we must show that any quasivariety \mathcal{Q} of R -modules is a variety.

Let Rd be the maximal ideal of R and let k be the least positive integer such that $d^k = 0$. Since LRFCRs are FGC rings, by Theorem 2.8 it suffices to show that if R/Rd^a is in \mathcal{Q} , then R/Rd^b is in \mathcal{Q} for any $b < a$. We may assume that \mathcal{Q} is defined by the single quasi-identity

$$\forall x_1, \dots, x_n \left[\bigwedge_{i=1}^k \left(\sum_{j=1}^n r_{ij} x_j = 0 \right) \rightarrow \sum_{j=1}^n s_j x_j = 0 \right].$$

Let $b < a$. Suppose for some $y_1, \dots, y_n \in R$, $\sum_j r_{ij} y_j \in Rd^b$, for all $i = 1, \dots, k$. That is, $\sum_j r_{ij} y_j = t_i d^b$ for some $t_i \in R$. Let $z_j = y_j d^{(a-b)}$, $j = 1, \dots, n$. Then

$$\sum_j r_{ij} z_j = \sum_j r_{ij} (y_j d^{(a-b)}) = \left(\sum_j r_{ij} y_j \right) d^{(a-b)} = t_i d^b d^{(a-b)} = t_i d^a \in Rd^a.$$

Since R/Rd^a is in \mathcal{Q} , this implies $\sum_j s_j z_j \in Rd^a$. So $\sum_j s_j z_j = t d^a$, for some t . $(\sum_j s_j y_j) d^{(a-b)} = t d^a$.

$$\left(\left(\sum_j s_j y_j \right) - t d^b \right) d^{(a-b)} = 0.$$

Therefore,

$$\left(\sum_j s_j y_j \right) - t d^b \in l(d^{(a-b)}) = Rd^{(k-(a-b))} = Rd^{b+(k-a)}.$$

Therefore, d^b divides $(\sum_j s_j y_j) - t d^b$, so d^b divides $(\sum_j s_j y_j)$, the conclusion of the quasi-identity is true, and $R/Rd^b \in \mathcal{Q}$. Therefore R is deductive.

Now suppose that R is an Artinian local ring which is not a left finite chain ring. We imitate the construction of Example 2.3: The maximal ideal N is finitely generated as an R -module. By Nakayama's Lemma [12, p. 413], $\{n_1, \dots, n_t\}$ is a minimal generating set for N as an R -module iff $\{n_1, \dots, n_t\}$ is a basis for N/N^2 over R/N . Since N is not principal, $t > 1$. Consider the quasivariety defined by

$$\forall x (n_1 x = 0 \rightarrow n_2 x = 0).$$

$R/N^2 \in \mathcal{Q}$, since if $n_1 r \in N^2$ then $r \in N$ (because if $r \notin N$ then r is a unit so n_1 would be in N^2 , which is a contradiction). Therefore $n_2 r \in N^2$. So $R/N^2 \in \mathcal{Q}$. But $R/(Rn_1 + N^2)$ is not in \mathcal{Q} , because $n_1 \bar{1} = \bar{0}$ but $n_2 \bar{1} \neq \bar{0}$. Thus \mathcal{Q} is not a variety and R is not deductive.

Finally, assume that R is a left finite chain ring which is not right chain. By replacing R with R/N^2 if necessary, we may assume $N^2 = (0)$. $dR \neq Rd$. Since $N = Rd$ is an ideal, $dR \subseteq Rd$. Therefore, there is an element ad of $Rd - dR$.

Let \mathcal{Q} be the quasivariety defined by

$$\forall x_1, x_2 (dx_1 + adx_2 = 0 \rightarrow dx_2 = 0).$$

R is in \mathcal{Q} : Suppose x_1 and x_2 satisfy the hypothesis. If x_2 were a unit then $ad = -dx_1 x_2^{-1}$, contradicting the choice of ad . Therefore $x_2 \in N$ and $dx_2 = 0$.

Since R is in \mathcal{Q} , if \mathcal{Q} were a variety it would be the variety of all R -modules. But this is not the case: In $R \oplus R$,

$$H = R(d, ad) = \{(ud, uad) : u \notin Rd\} \cup \{(0, 0)\}$$

is a submodule. Let $M = (R \oplus R)/H$. For $(r, s) \in R \oplus R$, let $\overline{(r, s)}$ denote its image in M . Then consider $x_1 = \overline{(1, 0)}$ and $x_2 = \overline{(0, 1)}$. $dx_1 + adx_2 = \overline{d(1, 0) + ad(0, 1)} = \overline{(d, ad)} = 0$. But $dx_2 = \overline{d(0, 1)} = \overline{(0, d)} \neq 0$. Therefore \mathcal{Q} is not a variety and R is not deductive. \square

2.16. COROLLARY. *Any proper homomorphic image of a principal ideal domain is deductive.*

PROOF. Any proper image of a PID is a commutative Artinian ring, hence the direct sum of local Artinian rings [15]. Since each of these local rings is the image of a PID, its maximal ideal is principal, i.e. it is an LRFCR. \square

The finiteness of the chain plays an important role in establishing deductiveness, as the following example shows:

2.17. EXAMPLE. Let R be the ring of formal power series over a field, $R = F[[x]]$. An element $f = \sum_{i=0}^{\infty} a_i x^i$ is a unit iff $a_0 \neq 0$. Every ideal of R is of the form Rx^i for some i . Thus R is a chain ring (the ideals of R form a chain), but R is not deductive, since it is an integral domain which is not a field.

Main theorem. The complete classification now follows from Theorems 2.4, 2.5, 2.9 and 2.15:

2.18. THEOREM. *An Artinian ring is deductive if and only if it is the direct sum of complete matrix rings over rings which are left and right finite chain rings.*

From this theorem, Lemma 2.14 and Theorem 2.9, we have

2.19. COROLLARY. *Every finitely generated module of a deductive Artinian ring is the direct sum of cyclic modules, i.e., a deductive Artinian ring is an FGC ring.*

Open question. Are all deductive rings direct products of matrix rings over LRFCRs?

3. Directly representable varieties. This section provides a characterization of deductiveness for a directly representable variety of finite type. Since the very definition of such a class is universal-algebraic in nature, one should expect the proof to employ techniques quite different from those of §2. However, we shall use R. McKenzie's powerful analysis of directly representable varieties [14] to reduce the question of deductiveness of the variety to that of a finite (hence Artinian) ring derived from the variety.

In the interest of brevity, we direct the reader to [2] for basic definitions and theorems. Our notation will be almost identical. Here we cover some additional concepts needed.

Let \mathcal{V} be a variety of algebras. The class of subdirectly irreducible algebras of \mathcal{V} is denoted \mathcal{V}_{SI} . \mathcal{V} is *residually finite* if every member of \mathcal{V}_{SI} is finite, and \mathcal{V} is *locally finite* if every finitely generated algebra in \mathcal{V} is finite. An element e of an algebra \mathbf{A}

is *idempotent* if $\{e\}$ is the universe of a subalgebra of \mathbf{A} . A variety has the *subalgebra condition* if every member has an idempotent.

Varieties (and quasivarieties) are ordered by inclusion. For a collection \mathcal{K} of algebras, $V(\mathcal{K})$ denotes the smallest variety containing \mathcal{K} , and $Q(\mathcal{K})$ the smallest quasivariety. The operators H , S , P , Pu and I denote closure under homomorphic images, subalgebras, products, ultraproducts and isomorphisms, respectively. Thus, a class \mathcal{K} is a variety iff $HSP(\mathcal{K}) = \mathcal{K}$ and is a quasivariety iff $ISPPu(\mathcal{K}) = \mathcal{K}$. A variety \mathcal{V} is *equationally complete* iff the only proper subvariety of \mathcal{V} is the variety of trivial algebras. Finally, an algebra $\mathbf{P} \in \mathcal{V}$ is *primitive* iff \mathbf{P} is finite, subdirectly irreducible and, for all $\mathbf{A} \in \mathcal{V}$, $\mathbf{P} \in H(\mathbf{A}) \Rightarrow \mathbf{P} \in IS(\mathbf{A})$.

We begin with some general observations about deductive varieties.

3.1. LEMMA. *Let \mathbf{A} and \mathbf{B} be nontrivial algebras and suppose $\mathbf{B} \in Q(\mathbf{A} \times \mathbf{B})$. Then there is a homomorphism from \mathbf{B} to an ultrapower of \mathbf{A} .*

PROOF. Recall $Q(\mathbf{A} \times \mathbf{B}) = ISPPu(\mathbf{A} \times \mathbf{B})$. It is easy to verify that an ultrapower $(\mathbf{A} \times \mathbf{B})^I/U$ is isomorphic to the product of the ultrapowers $\mathbf{A}^I/U \times \mathbf{B}^I/U$. Thus \mathbf{B} can be embedded into a product of the form $\prod_{j \in J} (\mathbf{A}^{I_j}/U_j \times \mathbf{B}^{I_j}/U_j)$. Since \mathbf{B} is nontrivial, $J \neq \emptyset$. Then for any $j \in J$, there is a projection of this product onto \mathbf{A}^{I_j}/U_j . Composing this projection with the embedding of \mathbf{B} yields the desired homomorphism. \square

3.2. THEOREM. *Let \mathcal{V} be a deductive variety of finite type. Then either \mathcal{V} has the subalgebra condition or no nontrivial member of \mathcal{V} has an idempotent.*

PROOF. Suppose an algebra \mathbf{A} of \mathcal{V} has no idempotent. Let \mathbf{B} be any nontrivial algebra of \mathcal{V} . We show that \mathbf{B} has no idempotent. Since \mathcal{V} is deductive, $Q(\mathbf{A} \times \mathbf{B})$ is closed under homomorphic images, hence contains \mathbf{B} . Therefore, by Lemma 3.1, there is a homomorphism from \mathbf{B} to an ultrapower \mathbf{A}^* of \mathbf{A} . Since the language of \mathcal{V} is finite, there is a first-order sentence Φ such that an algebra contains an idempotent if and only if it satisfies Φ . Now \mathbf{A} and \mathbf{A}^* satisfy the same first-order sentences and \mathbf{A} has no idempotent. Therefore \mathbf{A}^* , hence \mathbf{B} , has no idempotent. \square

In the special case that \mathcal{V} is semisimple, we can extract a little more information.

3.3. COROLLARY. *Let \mathcal{V} be a semisimple, deductive variety of finite type. Then either \mathcal{V} has the subalgebra condition, or \mathcal{V} is equationally complete.*

PROOF. Suppose $\mathcal{A} \in \mathcal{V}$ has no idempotent. Then by Theorem 3.2, no nontrivial member of \mathcal{V} has an idempotent. To show that \mathcal{V} is equationally complete it suffices to show that $\mathcal{V}_{SI} \subseteq V(\mathbf{B})$ for any nontrivial $\mathbf{B} \in \mathcal{V}$. For then $\mathcal{V} \subseteq V(\mathbf{B}) \subseteq \mathcal{V}$.

So let \mathbf{B} be a nontrivial member of \mathcal{V} , and \mathbf{C} a subdirectly irreducible algebra from \mathcal{V} . From the deductiveness of \mathcal{V} and Lemma 3.1, there is a homomorphism from \mathbf{C} to an ultrapower \mathbf{B}^* of \mathbf{B} . Since \mathbf{B}^* has no idempotent, the map is nontrivial, and since \mathbf{C} is simple, it must be one-to-one. Thus $\mathbf{C} \in IS(\mathbf{B}^*) \subseteq ISPu(\mathbf{B}) \subseteq V(\mathbf{B})$.

\square

The following proposition has been observed by a number of people.

3.4. THEOREM. *Let \mathcal{V} be residually finite and of finite type, or residually and locally finite. Then \mathcal{V} is deductive if and only if every subdirectly irreducible algebra in \mathcal{V} is primitive.*

PROOF. Suppose \mathcal{V} is deductive. Let \mathbf{B} be a subdirectly irreducible algebra from \mathcal{V} . To show \mathbf{B} is primitive, let $\mathbf{A} \in \mathcal{V}$ and suppose $\mathbf{B} \in H(\mathbf{A})$. By residual finiteness, \mathbf{B} is finite.

If the variety is of finite type, there is a first-order sentence Φ whose models are precisely the extensions of \mathbf{B} . Now $\mathbf{B} \in V(\mathbf{A}) = Q(\mathbf{A}) = \text{ISPPu}(\mathbf{A})$. Since \mathbf{B} is subdirectly irreducible, this implies that \mathbf{B} can be embedded into an ultrapower of \mathbf{A} , so this ultrapower satisfies Φ . Therefore \mathbf{A} itself satisfies Φ , so $\mathbf{B} \in \text{IS}(\mathbf{A})$.

If \mathcal{V} is locally finite and $\mathbf{B} \in H(\mathbf{A})$, then there is a finite subalgebra \mathbf{C} of \mathbf{A} such that $\mathbf{B} \in H(\mathbf{C})$. Applying the deductiveness of \mathcal{V} , $\mathbf{B} \in \text{ISPPu}(\mathbf{C}) = \text{ISP}(\mathbf{C})$ since \mathbf{C} is finite. Since \mathbf{B} is subdirectly irreducible, $\mathbf{B} \in \text{IS}(\mathbf{C}) \subseteq \text{IS}(\mathbf{A})$.

For the converse, suppose $\mathcal{Q} \subseteq \mathcal{V}$ is a quasivariety. To show \mathcal{Q} is closed under homomorphic images, let $\mathbf{A} \in \mathcal{Q}$, $\mathbf{C} \in H(\mathbf{A})$. Write \mathbf{C} as a subdirect product of subdirectly irreducible algebras from \mathcal{V} . As \mathcal{Q} is closed under subdirect products, it suffices to show that each of these subdirectly irreducibles is a member of \mathcal{Q} . But, if \mathbf{B} is such an algebra, then $\mathbf{B} \in H(\mathbf{C}) \subseteq H(\mathbf{A})$; so since \mathbf{B} is primitive, $\mathbf{B} \in \text{IS}(\mathbf{A}) \subseteq \mathcal{Q}$. \square

We now turn to abelian varieties. These have played a key role in the recent advances in the structure theory of congruence modular varieties. Here we give a quick review of the relevant details and encourage the reader to consult [6 or 7] for a thorough treatment.

A variety \mathcal{V} is *modular* iff the congruence lattice of every $\mathbf{A} \in \mathcal{V}$ is modular. An algebra \mathbf{A} is said to be *abelian* if and only if (1) every algebra in $V(\mathbf{A})$ has a modular congruence lattice, and (2) there is a congruence κ on $\mathbf{A} \times \mathbf{A}$ such that for every $x, y, z \in \mathbf{A}$: $\langle x, x \rangle \equiv \langle y, z \rangle \pmod{\kappa}$ iff $y = z$ (i.e. the diagonal of \mathbf{A} is a congruence class). This definition of abelian is different from the ones given in [6 and 7] (which are, in turn, different from each other) but all are equivalent when restricted to modular varieties. A variety is said to be abelian if every member is abelian. Such a variety will always be modular. For any modular variety \mathcal{V} , the collection of abelian algebras forms a subvariety \mathcal{V}_{ab} , which, of course, is abelian. An easy computation will show that a group is abelian (in this sense) if and only if it is commutative. A ring is abelian if and only if it satisfies the identity $xy = 0$. Any variety of modules over a ring is abelian and, in fact, the following construction shows that they constitute, in a sense, the most general abelian varieties.

Let \mathcal{V} be an arbitrary abelian variety. \mathcal{V} is actually *congruence permutable* (a stronger property than modularity), which implies that there is a term $d(x, y, z)$ in the language of \mathcal{V} such that $d(x, y, y) = d(y, y, x) = x$ is true in every algebra of \mathcal{V} . Furthermore, for every $\mathbf{A} \in \mathcal{V}$, the term d induces a homomorphism from \mathbf{A}^3 to \mathbf{A} . (The proof of all this is not easy. See [8, 7 or 17] for details.)

Now let $F_{\mathcal{V}}(u, v)$ be the \mathcal{V} -free algebra on two generators. The elements of this algebra can be thought of as terms $r(u, v)$ representing equivalence classes under the

relation: $\mathcal{V} \models r(u, v) = s(u, v)$. We construct a ring $R(\mathcal{V}) = (R, +, -, \cdot, 0, 1)$ as follows:

$$\begin{aligned} R &= \{ r(u, v) \in \mathbf{F}_{\mathcal{V}}(u, v) : \mathcal{V} \models r(x, x) = x \}, \\ 0 &= v, \quad 1 = u, \\ r(u, v) + s(u, v) &= d(r(u, v), v, s(u, v)), \\ -r(u, v) &= d(v, r(u, v), v), \\ r(u, v) \cdot s(u, v) &= r(s(u, v), v). \end{aligned}$$

Let \mathbf{A} be any algebra in \mathcal{V} , and z any member of A . We construct a left $R(\mathcal{V})$ -module $M(\mathbf{A}, z)$ as follows:

$$\begin{aligned} a + b &= d(a, z, b), \\ -a &= d(z, a, z), \\ 0 &= z, \\ r(u, v) \cdot a &= r(a, z) \quad \text{for any } r \in R. \end{aligned}$$

If z and z' are any two elements of A , then the map $\psi(x) = d(x, z, z')$ is a module isomorphism from $M(\mathbf{A}, z)$ to $M(\mathbf{A}, z')$. Furthermore, if z and z' are idempotents of \mathbf{A} , then ψ is an automorphism of \mathbf{A} as well.

Finally, define the module $N(\mathcal{V})$ to be $M(\mathbf{F}_{\mathcal{V}}(v), v)$, derived from the \mathcal{V} -free algebra on one generator. For any $R(\mathcal{V})$ -module M and any module homomorphism $\phi: N(\mathcal{V}) \rightarrow M$ we can construct a member of \mathcal{V} , denoted $A(M, \phi)$, with underlying set identical with that of M . If $G(x_1, \dots, x_n)$ is an operation symbol of \mathcal{V} , define elements r_1, \dots, r_n of $R(\mathcal{V})$ by

$$r_i(u, v) = d(G(v, \dots, u, \dots, v), G(v, \dots, v), v),$$

in which a lone u appears in the i th position of G . Then the operation interpreting G in $\mathbf{A}(M, \phi)$ is defined by

$$(1) \quad G(a_1, \dots, a_n) = \sum_{i=1}^n r_i a_i + \phi(G(v, \dots, v)) \quad \text{for any } a_1, \dots, a_n \in M.$$

To formalize the relationship between the two varieties, let (\mathcal{V}, p) denote the category whose objects are pointed \mathcal{V} -algebras (\mathbf{A}, z) , $z \in A$, and morphisms consisting of all \mathcal{V} -homomorphisms preserving the specified points. Let $(N \downarrow R)$ denote the “comma category” of all pairs (M, ϕ) such that M is a left R -module and ϕ is a module homomorphism from N to M . Morphisms from (M_1, ϕ_1) to (M_2, ϕ_2) are R -module homomorphisms $\psi: M_1 \rightarrow M_2$ such that $\psi \circ \phi_1 = \phi_2$. Then we have the following

3.5. THEOREM [6, THEOREM 9.9]. *Let \mathcal{V} be an abelian variety. The categories (\mathcal{V}, p) and $(N(\mathcal{V}) \downarrow R(\mathcal{V}))$ are isomorphic. To each pointed algebra (\mathbf{A}, z) the isomorphism assigns the pair $(M(\mathbf{A}, z), \phi)$ in which $\phi(v) = z$. To each pair (M, ϕ) , the inverse*

isomorphism assigns $(A(M, \phi), 0)$. The structures \mathbf{A} and $M(\mathbf{A}, z)$ have the same polynomial functions and the same congruences.

3.6. EXAMPLE. A quasigroup $\mathbf{Q} = (Q, \circ, /, \backslash)$ is a structure with 3 binary operations satisfying the identities

$$(2) \quad \begin{aligned} x \circ (x \backslash y) &= y, & x \backslash (x \circ y) &= y, \\ (y/x) \circ x &= y, & (y \circ x)/x &= y. \end{aligned}$$

The variety of all quasigroups is a modular variety, in fact permutable. Let $d(x, y, z) = (x/y) \circ [(y/y) \backslash z]$. Then the identities $d(x, x, y) = d(y, x, x) = y$ follow from (2). We isolate an interesting abelian subvariety defined by the additional three identities

$$\begin{aligned} (c) \quad & x \circ y = y \circ x, \\ (i) \quad & x \circ x = x, \\ (a) \quad & (x \circ y) \circ (w \circ z) = (x \circ w) \circ (y \circ z). \end{aligned}$$

Following Ježek and Kepka [13], we call this subvariety the variety of cia-quasigroups (commutative, idempotent, abelian quasigroups—any quasigroup satisfying (a) is abelian). Observe that every cia-quasigroup also satisfies: $x/y = y \backslash x$ and $x/x = x$. To see it is abelian, let \mathbf{Q} be a cia-quasigroup, and define κ on $\mathbf{Q} \times \mathbf{Q}$ by $\langle x_1, x_2 \rangle \equiv \langle y_1, y_2 \rangle \pmod{\kappa} \Leftrightarrow x_1 \circ y_2 = y_1 \circ x_2$. It is a straightforward verification that κ is a congruence on $\mathbf{Q} \times \mathbf{Q}$ and that the diagonal is a congruence class.

To define module addition and subtraction, choose any $z \in Q$ to be the zero element. Then

$$a + b = d(a, z, b) = (a/z) \circ [(z/z) \backslash b] = (a/z) \circ (z \backslash b) = (a \circ b)/z$$

and

$$-a = d(z, a, z) = (z/a) \circ [(a/a) \backslash z] = z/a.$$

(This has been known since 1940 as a special case of “Toyoda’s Theorem” [18].)

To describe the free algebra, let F be the set of all rational numbers of the form $a \cdot 2^{-k}$, a an integer, k a natural number. For $b, c \in F$, let $b \circ c = \frac{1}{2}(b + c)$, $b/c = c \backslash b = 2b - c$. Then the free cia-quasigroup on generators u and v is isomorphic to $\mathbf{F} = (F, \circ, /, \backslash)$ in which u is mapped to 1 and v to 0 (see [13]). To produce the ring $R(\mathcal{V})$, note that for every term $r(u, v)$, $\mathcal{V} \models r(x, x) = x$, so the universe of R is identical with that of the free algebra. Now define $r(u, v) + s(u, v) = d(r, v, s) = (r \circ s)/v$. But under the isomorphism with the algebra \mathbf{F} , $(r \circ s)/v$ is mapped to the sum of the images of r and s . Similarly, the product in the ring corresponds to the product of rational numbers in F (use induction on the length of a term). Thus we see that the ring $R(\mathcal{V}) \cong \mathbf{Z}[\frac{1}{2}]$. The ring element $\frac{1}{2}$ corresponds to the term $(u \circ v)$ and, in any module $M(\mathbf{Q}, z)$, $\frac{1}{2}x = (x \circ z)$. Conversely, for any module M , we can retrieve the cia-quasigroup by defining $b \circ c$ to be $\frac{1}{2}(b + c)$ (thus, in (1), $r_1 = r_2 = \frac{1}{2}$ and $\phi(G(v, \dots, v)) = 0$).

We would now like to apply Theorem 3.5 to reduce the question of deductiveness of an arbitrary abelian variety \mathcal{V} to that of the ring $R(\mathcal{V})$. The key is to show that the information contained in the module $N(\mathcal{V})$ is not needed. This will follow from Theorem 3.2.

3.7. THEOREM. *Let \mathcal{V} be an abelian variety of finite type. Then \mathcal{V} is deductive if and only if \mathcal{V} has the subalgebra condition and $R(\mathcal{V})$ is deductive.*

PROOF. Let \mathbf{A} be any nontrivial member of \mathcal{V} and let κ be the congruence on $\mathbf{A} \times \mathbf{A}$ that contains the diagonal as a class. Then $\mathbf{B} = (\mathbf{A} \times \mathbf{A})/\kappa$ is a nontrivial member of \mathcal{V} containing an idempotent (namely the diagonal). Thus by Theorem 3.2, the subalgebra condition is necessary.

Now suppose the pointed \mathcal{V} -algebra (\mathbf{A}, z) corresponds to the pair (M, ϕ) under the isomorphism of 3.5. The element z is an idempotent of \mathbf{A} if and only if the map ϕ is trivial. Thus, by restricting ourselves to idempotent elements, the pairs $M(\mathbf{A}, z)$ will be all $R(\mathcal{V})$ -modules together with trivial maps (which can be ignored). Conversely, starting with an arbitrary module M and the trivial map $\mathbf{0}: N(\mathcal{V}) \rightarrow M$, the pointed algebras $\mathbf{A}(M, \mathbf{0})$ will be all members of \mathcal{V} together with all idempotents.

Let \mathcal{Q} be a subquasivariety of \mathcal{V} . Define \mathcal{Q}_M to be the class of all $M(\mathbf{A}, z)$ such that $\mathbf{A} \in \mathcal{Q}$ and $z \in \mathbf{A}$ is idempotent. It is a somewhat tedious, but not difficult, exercise to show that \mathcal{Q}_M is closed under the formation of submodules, products and ultraproducts (use the corresponding property of \mathcal{Q} together with 3.5 and the fact that if z and z' are idempotents of \mathbf{A} , then $(\mathbf{A}, z) \simeq (\mathbf{A}, z')$ and $M(\mathbf{A}, z) \simeq M(\mathbf{A}, z')$). Thus \mathcal{Q}_M is a quasivariety of $R(\mathcal{V})$ -modules. Furthermore, \mathcal{Q} is closed under homomorphic images if and only if \mathcal{Q}_M is so closed. Therefore, if $R(\mathcal{V})$ is deductive, \mathcal{Q}_M is a variety, \mathcal{Q} is a variety, so \mathcal{V} is deductive.

Conversely, given a quasivariety \mathcal{Q} of $R(\mathcal{V})$ -modules, let $\mathcal{Q}_{\mathcal{V}}$ be all algebras $\mathbf{A} \in \mathcal{V}$ such that for some idempotent $z \in \mathbf{A}$, $M(\mathbf{A}, z) \in \mathcal{Q}$ (equivalently, $(\mathbf{A}, z) = \mathbf{A}(M, \mathbf{0})$ for some $M \in \mathcal{Q}$). Using a similar line of reasoning, $\mathcal{Q}_{\mathcal{V}}$ is a quasivariety, and the deductiveness of \mathcal{V} implies that of $R(\mathcal{V})$. \square

Returning to cia-quasigroups, let us apply Theorem 3.7. The variety certainly has the subalgebra condition (by identity (i)) so \mathcal{V} is deductive iff the ring $\mathbf{Z}[\frac{1}{2}]$ is deductive. The ring is not Artinian, but as remarked earlier, (Example 2.2) the ring cannot be deductive since it is an integral domain which is not a field. On the other hand, $\mathbf{Z}[\frac{1}{2}]$ is a PID and, by 2.16, every proper homomorphic image will be deductive. Translating back to quasigroups, every proper subvariety of cia-quasigroups is a deductive variety.

We now apply Theorems 2.18 and 3.7 to characterize deductiveness for an important class of varieties. A variety \mathcal{V} is *directly representable* if there is a finite set \mathcal{X} of finite algebras such that $\mathcal{V} = V(\mathcal{X})$ and every finite algebra of \mathcal{V} is a member of $P(\mathcal{X})$. As an example: for any natural number n , the variety of abelian groups of exponent n is directly representable.

For an algebra \mathbf{A} in a modular variety define the congruence

$$[1, 1] = \cap \{ \beta \in \text{Con}(\mathbf{A}) : \mathbf{A}/\beta \text{ is abelian} \}.$$

Since the collection of abelian algebras forms a subvariety, $\mathbf{A}/[1, 1]$ is abelian.

We require the following properties of directly representable varieties from McKenzie [14, Theorems 5.4, 5.9, 5.11].

Let \mathcal{V} be a directly representable variety. Then \mathcal{V} is permutable (hence modular) and every subdirectly irreducible algebra in \mathcal{V} is finite and either simple or abelian. For every finite algebra $\mathbf{A} \in \mathcal{V}$, there is a congruence ξ on \mathbf{A} such that:

- (1) $\mathbf{A} \cong \mathbf{A}/\xi \times \mathbf{A}/[1, 1]$ (under the canonical homomorphism).
- (2) $\mathbf{A}/\xi \cong \mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_n$, each \mathbf{B}_i simple and nonabelian, and this representation is unique (up to the order of the factors).
- (3) For every $\alpha, \beta \in \text{Con}(\mathbf{A})$, $\xi \cap (\alpha \vee \beta) = (\xi \cap \alpha) \vee (\xi \cap \beta)$ (i.e. ξ is “neutral”).

Thus we have the following characterization.

3.8. THEOREM. Let \mathcal{V} be a directly representable variety of finite type. \mathcal{V} is deductive if and only if:

- (1) \mathcal{V} is equationally complete, or
- (2) \mathcal{V} has the subalgebra condition and \mathcal{V}_{ab} is deductive. Furthermore, \mathcal{V}_{ab} is deductive if and only if $R(\mathcal{V}_{ab})$ is the direct sum of matrix rings over LRFCRs.

PROOF. Let \mathcal{V} be deductive. Suppose first that \mathcal{V} fails the subalgebra condition. Then by 3.2 no nontrivial algebra in \mathcal{V} has an idempotent. If $\mathbf{A} \in \mathcal{V}$ is abelian, $(\mathbf{A} \times \mathbf{A})/\kappa$ has an idempotent. Thus \mathcal{V} has no nontrivial abelian algebras, so \mathcal{V} must be semisimple. By Corollary 3.3, \mathcal{V} is equationally complete. On the other hand, if \mathcal{V} has the subalgebra condition, then alternative (2) obviously holds.

For the converse, let \mathcal{V} be equationally complete. Let \mathbf{M} be a finite, nontrivial algebra of \mathcal{V} of smallest cardinality. (Such an algebra exists because \mathcal{V} is locally finite.) \mathbf{M} is simple, has no proper, nontrivial subalgebras and generates \mathcal{V} , which is permutable. Therefore by [16, Theorems 1.1, and 1.2], every finite algebra in \mathcal{V} is a direct power of \mathbf{M} , so \mathbf{M} is the only subdirectly irreducible algebra in \mathcal{V} . Since \mathbf{M} can be embedded into every direct power of \mathbf{M} , and since \mathcal{V} is locally finite, \mathbf{M} is primitive. Therefore by Theorem 3.4, \mathcal{V} is deductive.

Now suppose \mathcal{V} has the subalgebra condition and \mathcal{V}_{ab} is deductive. By Theorem 3.4 it is sufficient (and necessary) to show that every subdirectly irreducible algebra in \mathcal{V} is primitive. Let \mathbf{B} be subdirectly irreducible in \mathcal{V} and suppose $\mathbf{B} = \mathbf{A}/\theta$ for some $\mathbf{A} \in \mathcal{V}$. Since \mathcal{V} is locally finite, we may assume \mathbf{A} is finite. There are two cases.

Case 1. \mathbf{B} is abelian.

Then $\theta \supseteq [1, 1]$ in $\text{Con}(\mathbf{A})$. Let $\mathbf{C} = \mathbf{A}/[1, 1]$. Then $\mathbf{B} \in H(\mathbf{C})$ and $\mathbf{C} \in \mathcal{V}_{ab}$. By assumption, \mathcal{V}_{ab} is deductive, so $\mathbf{B} \in \text{IS}(\mathbf{C})$ by 3.4. But $\mathbf{A} \cong \mathbf{A}/\xi \times \mathbf{A}/[1, 1] = \mathbf{A}/\xi \times \mathbf{C}$. Since \mathbf{A}/ξ has an idempotent, \mathbf{B} can be embedded into the right-hand product, hence $\mathbf{B} \in \text{IS}(\mathbf{A})$.

Case 2. \mathbf{B} is nonabelian.

Then \mathbf{B} is simple, so θ is a coatom of $\text{Con}(\mathbf{A})$. Furthermore, $\theta \not\supseteq [1, 1]$, so $\theta \vee [1, 1] = 1$ (the largest congruence of \mathbf{A}). Since ξ is neutral, $\xi = \xi \cap (\theta \vee [1, 1]) = (\xi \cap \theta) \vee (\xi \cap [1, 1]) = \xi \cap \theta$, in other words, $\xi \subseteq \theta$. Since \mathbf{A}/ξ is uniquely expressible as a product of simple algebras, there is an algebra \mathbf{B}' such that $\mathbf{A}/\xi \cong \mathbf{B} \times \mathbf{B}'$. Therefore, $\mathbf{A} \cong \mathbf{B} \times \mathbf{B}' \times \mathbf{A}/[1, 1]$ and from the subalgebra condition, we conclude that $\mathbf{B} \in \text{IS}(\mathbf{A})$.

The final claim about $R(\mathcal{V}_{ab})$ follows from Theorems 2.18 and 3.7 since the ring $R(\mathcal{V})$ is finite, so certainly Artinian. \square

EXAMPLE. Let \mathcal{V} be the variety generated by the finite field $\mathbf{GF}(p^k)$, p a prime and k a positive integer. \mathcal{V} is directly representable. If $k = 1$, \mathcal{V} is equationally complete, and is therefore deductive. For $k > 1$, \mathcal{V} is not equationally complete, and fails the subalgebra condition since $0 \neq 1$. Thus \mathcal{V} is not deductive.

Let $\mathbf{Q} = (\{0, 1, 2, 3, 4\}, \circ, /, \backslash)$ be the quasigroup with Cayley table

\circ	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	3	4	0	1
3	3	4	1	2	0
4	4	2	0	1	3

and let $\mathcal{V} = V(\mathbf{Q})$. Since the order of \mathbf{Q} is prime, \mathbf{Q} is simple.

\mathbf{Q} has a single subalgebra, \mathbf{A} with universe $\{0, 1\}$. \mathbf{A} is isomorphic to the group \mathbf{Z}_2 , so \mathbf{A} is simple (and abelian). By [3, Theorem 1.6, 1.7], \mathcal{V} is directly representable by $\{\mathbf{Q}, \mathbf{A}\}$ and has no other subdirectly irreducible members. \mathbf{Q} is nonabelian (since it is simple, but has a subalgebra) and \mathcal{V} has the subalgebra condition (since \mathbf{Q} and \mathbf{A} have idempotents). The subvariety \mathcal{V}_{ab} is generated by \mathbf{A} , so the associated ring is \mathbf{Z}_2 . By Theorem 2.18, \mathcal{V}_{ab} is deductive and, by 3.8, \mathcal{V} is deductive.

3.9. COROLLARY. *Every finitely generated, abelian, deductive variety of finite type is directly representable.*

PROOF. Let R be the ring associated with the variety. It suffices to show that the variety of left R -modules is directly representable. By 3.7, R is deductive and, by 2.19, R is an FGC ring. Since R is finite it has only finitely many cyclic modules, and these are the only directly indecomposables. Thus, the variety is directly representable. \square

In light of the corollary, one might wonder whether every finitely generated, deductive, permutable variety is directly representable. Here is a counterexample.

Let \mathbf{A} be the 3-element Heyting algebra, and $\mathcal{V} = V(\mathbf{A})$. \mathcal{V} is both distributive and permutable. \mathbf{A} is subdirectly irreducible and has a single proper homomorphic image \mathbf{B} , the 2-element Heyting algebra. Thus \mathbf{A} and \mathbf{B} are the only two subdirectly irreducible algebras of \mathcal{V} . For \mathcal{V} to be deductive, we need both \mathbf{A} and \mathbf{B} to be primitive (3.4). However, \mathbf{B} is a subalgebra of every nontrivial member of \mathcal{V} , and \mathbf{A} is projective in this variety, so certainly primitive. On the other hand, \mathcal{V} is not directly representable since \mathbf{A} is neither simple nor abelian.

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DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50011